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FUNDAMENTAL SOLUTIONS OF THERMOELASTICITY EQUATIONS FOR A SEMIMOMENT ISOTROPIC MEDIUM

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The Kupradze matrix of fundamental solution of the system of thermoelasticity equations for a semimoment isotropic medium with account for the relaxation time of thermal perturbations in it has been obtained.

Introduction. In solving problems (including boundary ones) for a wide range of systems of second-order, partial differential linear equations in three-dimensional elasticity theory, the method of potentials and the theory of integral equations are widely used [1–3]. In the majority of cases, potentials are determined from the fundamental of other singular solutions of corresponding differential equations [4]. The construction of fundamental solutions of equations for homogeneous and inhomogeneous isotropic or anisotropic media with or without regard for the couple stresses in them on the basis of algebraic methods was described in sufficient detail in [4, 5]. In the present work, this important direction of investigations was developed for semimoment, isotropic, thermoelastic medium, the thermal properties of which are described by a hyperbolic heat-conduction equation.

Matrix Differential Operator. The system of equations for the generalized interrelated dynamic problem on the thermoelasticity of a semimoment isotropic medium in the Cartesian coordinate system can be represented in the following form [6–8]:

$$(\lambda + \mu + \eta\Delta)\,\mathrm{grad\,div}\,\mathbf{u} + (\mu - \eta\Delta)\,\Delta\mathbf{u} + \frac{\rho}{2}\,\mathrm{rot}\,\mathbf{l} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}} + \beta\,\mathrm{grad}\,\theta\,,$$
$$K\Delta\theta - c_\varepsilon\,(\dot{\theta} + \tau\ddot{\theta}) = \beta T_0\,(\mathrm{div}\,\dot{\mathbf{u}} + \tau\,\mathrm{div}\,\ddot{\mathbf{u}}) - q\,. \tag{1}$$

It is assumed that external forces acting on the indicated medium (body forces **f**, volume moments **l**, internal heat sources q) oscillate with a frequency ω. In this case, from (1) we obtain the following equation for the thermoelastic vibrational state of this medium:

$$(\lambda + \mu + \eta\Delta)\,\mathrm{grad\,div}\,\mathbf{u}_k + (\mu - \eta\Delta)\,\Delta u_k + \rho\omega^2 u_k - \beta\,\mathrm{grad}\,\theta = 0\,,\quad k = \overline{1,3}\,,$$
$$K\Delta\theta + c_\varepsilon\,(i\omega + \tau\omega^2)\theta + \beta T_0\,(i\omega + \tau\omega^2)\,\mathrm{div}\,\mathbf{u} + q = 0\,. \tag{2}$$

Let us introduce the matrix differential operator

$$M = \left\| M_{kl} \right\|_{4\times4}\,. \tag{3}$$

Here, $M_{jn} = ((\lambda + \mu) + \eta\Delta)\partial_j\partial_n + ((\mu - \eta\Delta)\Delta + \rho\omega^2)\delta_{jn}$, where $n, j = \overline{1, 3}$; $M_{4n} = (i + \tau\omega)\omega T_0\beta\partial_n$; $M_{n4} = -\beta\partial_n$; and $M_{44} = K\Delta + (i + \tau\omega)c_\varepsilon\omega$.

The equation for the thermoelastic vibrational state of the semimoment thermoelastic medium being considered can be written with account for the relaxation time of thermal perturbations and corresponding **f**, **l**, and q in the following form:

$$MU = 0\,, \tag{4}$$

where M is the matrix differential operator (3). The determinant of the matrix M is equal to

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$$\det \| M \| = \mu^2 (\lambda + 2\mu) \left(\left(1 - \frac{\eta\Delta}{\mu} \right) \Delta + k_2^2 \right)^2 \left(\Delta + \beta_1^2 \right) \left(\Delta + \beta_2^2 \right). \tag{5}$$

We now determine the algebraic complement m_{kl} of the element M_{kl} in the determinant $\det \|M\|$. After simple transformations, we obtain

$$\begin{gathered}
m_{nn} = \mu \left(\Delta - \frac{\eta}{\mu} \Delta^2 + k_2^2 \right) \left(T_0 \omega \beta^2 (i + \omega\tau) \left(\Delta - \partial_n^2 \right) + (c_\varepsilon \omega (i + \omega\tau) + K\Delta) \right) \times \\
\times \left(\left(\mu - \eta \left(\Delta - \partial_n^2 \right) \right) \partial_n^2 - \eta \partial_n^4 + (\lambda + 2\mu) \left(\Delta - \partial_n^2 \right) + \omega^2 \rho \right), \\
m_{nj} = - \mu \left(T_0 \omega \beta^2 + (c_\varepsilon \omega (i + \omega\tau) + K\Delta) (\lambda + \mu + \eta\Delta) \right) \left(\Delta - \frac{\eta}{\mu} \Delta^2 + k_2^2 \right) \partial_n \partial_j , \\
m_{n4} = - (i + \omega\tau) T_0 \omega \beta \mu^2 \left(\Delta - \frac{\eta}{\mu} \Delta^2 + k_2^2 \right)^2 \partial_n , \\
m_{44} = \mu^2 (\lambda + 2\mu) \left(\Delta + \frac{\omega^2 \rho}{(\lambda + 2\mu)} \right) \left(\Delta - \frac{\eta}{\mu} \Delta^2 + k_2^2 \right)^2 , \\
m_{4n} = \beta \mu^2 \left(\Delta - \frac{\eta}{\mu} \Delta^2 + k_2^2 \right)^2 \partial_n , \quad n \neq j , \quad n, j = \overline{1, 3} .
\end{gathered} \tag{6}$$

Let us express the column vector U in terms of the matrix differential operator $\hat{m}$ obtained by transposing of m:

$$U = \hat{m} \varphi . \tag{7}$$

Substitution of U into (4) gives

$$M \hat{m} \varphi = \left\| \mu^2 (\lambda + 2\mu) \delta_{kl} \left(\left(1 - \frac{\eta\Delta}{\mu} \right) \Delta + k_2^2 \right)^2 \left(\Delta + \beta_1^2 \right) \left(\Delta + \beta_2^2 \right) \varphi \right\| = 0 , \quad j, k = \overline{1, 4} . \tag{8}$$

From relation (8) defining the function φ follows the equation

$$\left(\left(1 - \frac{\eta\Delta}{\mu} \right) \Delta + k_2^2 \right)^2 \left(\Delta + \beta_1^2 \right) \left(\Delta + \beta_2^2 \right) \varphi = 0 .$$

Since all the elements of the matrix $\hat{m}$ include the multiplier $\left(\left(1 - \frac{\eta\Delta}{\mu} \right) \Delta + k_2^2 \right) \varphi$, it will be sufficient to determine the function ψ in the following form:

$$\psi = \mu^2 (\lambda + 2\mu) \left(\left(1 - \frac{\eta\Delta}{\mu} \right) \Delta + k_2^2 \right) \varphi . \tag{9}$$

For this purpose, we will use the equation

$$\left(\Delta + \beta_1^2 \right) \left(\Delta + \beta_2^2 \right) \left(\Delta + \beta_3^2 \right) \left(\Delta + \beta_4^2 \right) \psi = 0 . \tag{10}$$

Matrix of Fundamental Solutions. Let us find the solution of (10) for which the six-order partial derivatives will have singularities of the form $|x|^{-1} = (x_1^2 + x_2^2 + x_3^2)^{-1/2}$. The desired solution should satisfy the following conditions:

$$\left(\Delta+\beta_k^2\right)\psi=\frac{\gamma}{|x|}\left(\exp\left(i\beta_{k+1}\ |x|\right)\left(\beta_{k+2}^2-\beta_{k+3}^2\right)+\right.$$

$$\left.+\exp\left(i\beta_{k+2}\ |x|\right)\left(\beta_{k+3}^2-\beta_{k+1}^2\right)+\exp\left(i\beta_{k+3}\ |x|\right)\left(\beta_{k+1}^2-\beta_{k+2}^2\right)\right),$$

where $\gamma = (2\pi(\beta_{k+2}^2-\beta_{k+3}^2)(\beta_{k+1}^2-\beta_{k+3}^2)(\beta_{k+1}^2-\beta_{k+2}^2))^{-1}$, $\beta_5\equiv\beta_1$, $\beta_6\equiv\beta_2$, $\beta_7\equiv\beta_3$, and $k = \overline{1,4}$. Hence it follows that the function ψ is equal to

$$\psi=\frac{1}{2\pi\ |x|}\sum_{k=1}^{4}\frac{\exp\left(i\beta_k\ |x|\right)}{\left(\beta_{k+1}^2-\beta_k^2\right)\left(\beta_{k+2}^2-\beta_k^2\right)\left(\beta_{k+3}^2-\beta_k^2\right)}. \tag{11}$$

Substitution of the function ψ into (7) with account for the expression

$$\left(\left(1-\frac{\eta\Delta}{\mu}\right)\Delta+k_2^2\right)\varphi=\frac{\psi}{\mu^2\,(\lambda+2\mu)} \tag{12}$$

gives, after simple transformations, the matrix of fundamental solutions of the system of equations

$$\Gamma\,(x,\omega)=\left\|\ \Gamma_{kl}\,(x,\omega)\right\|_{4\times 4}\ . \tag{13}$$

Here

$$\Gamma_{jj}=-\gamma\sum_{k=1}^{4}\frac{\exp\,(i\ |x|\ \beta_k)}{|x|}\left(c_\varepsilon\omega\,(i+\tau\omega)\left(\left(\lambda+\mu-\eta\beta_k^2\right)\times\right.\right.$$

$$\left.\times\frac{1-i\ |x|\ \beta_k}{|x|^2}\,\frac{3x_j^2-|x|^2}{|x|^2}+\beta_k^2\left(\lambda+\ 2\mu-\left(\lambda+\mu-\eta\beta_k^2\right)\frac{x_j^2}{|x|^2}\right)\right)+$$

$$+T_0\omega\beta^2\,(i+\tau\omega)\left(\frac{1-i\ |x|\ \beta_k}{|x|^2}\,\frac{3x_j^2-|x|^2}{|x|^2}+\beta_k^2\,\frac{|x|^2-x_j^2}{|x|^2}\right)-$$

$$-K\beta_k^2\left((\lambda+\mu)\,\frac{1-i\ |x|\ \beta_k}{|x|^2}\,\frac{3x_j^2-|x|^2}{|x|^2}+\right.$$

$$\left.\left.+\beta_k^2\left(\lambda+2\mu-\left(\lambda+\mu-\eta\beta_k^2\right)\frac{x_j^2}{|x|^2}-\eta\,\frac{1-i\ |x|\ \beta_k}{|x|^2}\,\frac{3x_j^2-|x|^2}{|x|^2}\right)\right)\right);$$

$$\Gamma_{jn}=\gamma\sum_{k=1}^{4}\frac{\exp\,(i\ |x|\ \beta_k)}{|x|}\,\frac{x_jx_n}{|x|^2}\left(\beta_k^2-\frac{3\,(1-i\ |x|\ \beta_k)}{|x|^2}\right)\times$$

$$\times\left(\omega\,(i+\omega\tau)\left(T_0\beta^2+c_\varepsilon\,(\alpha+\mu)\right)+K\eta\beta_k^4-\left(K\,(\lambda+\mu)+c_\varepsilon\omega\eta\,(i+\omega\tau)\right)\beta_k^2\right);$$

$$\Gamma_{j4}=\gamma\sum_{k=1}^{4}\frac{\exp\,(i\ |x|\ \beta_k)}{|x|}\,\frac{(1-i\ |x|\ \beta_k)}{|x|}\,\frac{x_j\beta\left(\beta_k^2\left(\mu+\beta_k^2\lambda\right)-\omega^2\rho\right)}{|x|};$$

$$\Gamma_{4j} = -\gamma \sum_{k=1}^{4} \frac{\exp(i\,|x|\,\beta_k)}{|x|} \frac{(1 - i\,|x|\,\beta_k)}{|x|} \frac{x_j T_0 \omega\beta\,(1 + i\tau)\left(\beta_k^2\left(\mu + \beta_k^2\lambda\right) - \omega^2\rho\right)}{|x|};$$

$$\Gamma_{44} = \frac{\gamma \exp(i\,|x|\,\beta_k)}{|x|}\left(\omega^2\rho - (\lambda + 2\mu)\,\beta_k^2\right)\left(\omega^2\rho - \left(\mu + \eta\beta_k^2\right)\beta_k^2\right);$$

$$\gamma = -\left(2\pi\left(\beta_{k+1}^2 - \beta_k^2\right)\left(\beta_{k+2}^2 - \beta_k^2\right)\left(\beta_{k+3}^2 - \beta_k^2\right)\right)^{-1}.$$

Each column of matrix (13), considered as a vector, fits system (4) at all the points of the space E_3, except for the origin of coordinates.

Conclusions. Formulas (13) can be used to describe of the thermoelastic vibrational state of a semimoment isotropic medium with account for the time of relaxation of thermal perturbations. At $\tau = 0$, the fundamental matrix (13) for the system of hyperbolic-thermoelasticity equations of a semimoment medium is transformed into the Kupradze matrix for the thermoelastic semimoment medium, the thermal properties of which are defined by the classical thermoelasticity law.

NOTATION

c_ε, heat capacity at a constant deformation; i, imaginary unit; $k_2^2 = \rho\omega^2/\mu$; K, heat-conductivity coefficient; q, quantity of heat produced in a unit volume for a unit time; T_0, initial temperature; $\mathbf{u} = (u_1, u_2, u_3)$, displacement vector; $U = (u_1, u_2, u_3, \theta)$, four-component vector-column; α_t, coefficient of thermal expansion; β, constant relating the mechanical and thermal stresses, $\beta = \alpha_t(3\lambda + 2\mu)$; $\beta_1^2 + \beta_2^2 = -\left(\frac{(i + \tau\omega)T_0\omega\beta^2}{K(\lambda + 2\mu)} + \frac{(i + \tau\omega)c_\varepsilon\omega}{K} + \frac{\rho\omega^2}{\lambda + 2\mu}\right)$; $\beta_1^2\beta_2^2 = \frac{ic_\varepsilon\omega^3\rho(i + \tau\omega)}{K(\lambda + 2\mu)}$; $\beta_3^2 + \beta_4^2 = -\frac{\mu}{\eta}$; $\beta_3^2\beta_4^2 = -\frac{\mu k_2^2}{\eta}$; δ_{jn}, Kronecker symbol; Δ, Laplace operator; η, micropolar elastic constant; φ, scalar function; λ, μ, Lamé constants; θ, temperature change; ρ, density of a medium; τ, relaxation of thermal perturbations; $\partial_n = \partial/\partial x_n$; ω, oscillation frequency; point, differentiation with respect to time. Subscripts: t, thermal.

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